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Analytic solution for axisymmetric flow and heat transfer of a second grade fluid past a stretching sheet

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Abstract

The steady laminar flow and heat transfer of a second grade fluid over a radially stretching sheet is considered. The axisymmetric flow of a second grade fluid is induced due to linear stretching of a sheet. The heat transfer analysis has been carried out for two heating processes, namely (i) with prescribed surface temperature (PST-case) and (ii) prescribed surface heat flux (PHF-case). Introducing the dimensionless quantities the governing partial differential equations are transformed to ordinary differential equations. The developed non-linear differential equations are solved analytically using homotopy analysis method (HAM). The series solutions are developed and the convergence of these solutions is explicitly discussed. The analytical expressions for velocity and temperature are constructed and are shown graphically. The numerical values for the skin friction coefficient and the Nusselt number are entered in tabular form. Attention has been focused to the variations of the emerging parameters such as second grade parameter, Prandtl number and the Eckert number. Finally, comparison between the HAM and numerical solutions are also included and found in excellent agreement.

Keywords: Heat transfer; Second grade fluid; Axisymmetric flow; HAM solution

1. Introduction

In recent years, the viscoelastic fluids are recognized more appropriate than Newtonian fluids. This is due to their many practical applications in petroleum drilling, manufacturing of foods and paper and many other activities. Specifically, the boundary layer concept of viscoelastic fluids is of special importance owing to its application in many engineering problems, among which we cite the possibility of reducing frictional drag on the hulls of ships and submarines. Further, flow and heat transfer phenomena over stretching surface has promising applications in a number of technological processes including production of polymer films or thin sheets. Because of the non-linear nature of the dependence of stresses on the rate of strain for viscoelastic fluids, the solutions of flow problems for these fluids are in general more difficult to obtain. This is not only true of analytical solutions but even of numerical solutions. Due to these facts, the flows of viscoelastic fluids have been a challenging research topic for mathematicians, physicists and engineers alike.

A literature survey indicated that there has been an extensive literature available regarding the boundary layer flow over a planer stretching sheet in various situations. Such studies include different fluid models, magnetohydrodynamic and hydrodynamic cases, with and without heat transfer analysis. As can be seen from the literature, there has been relatively scarce information regarding the axisymmetric flow over a radial stretching sheet. Recently, Ariel [1] discussed the axisymmetric flow of a second grade fluid past a stretching sheet. He found exact, perturbative, asymptotic and numerical solutions for the problem. To the best of our information not a single attempt is available in the literature which describes the axisymmetric flow and

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heat transfer analysis over a radial stretching sheet. This study will help to fill this gap. The layout of the paper has been organized as follows:

We start our formulation in Section 2 by defining the continuity, momentum, constitutive equations and boundary conditions in the cylindrical coordinates. In Section 2.1, we find the analytic solution for the velocity using HAM [5–21]. The expression for the skin friction coefficient is also given in Section 2.2. The energy equation for the thermodynamic second grade fluid is presented in Section 3. Sections 3.1 and 3.2 respectively deal with the boundary conditions and HAM solutions of the temperature distribution for the prescribed surface temperature case and the prescribed heat flux case. In Section 4, we show the convergence of the solution and the behaviour of the convergence on the emerging parameters. In Section 5 the results relevant to the graphs are presented beside the comparison of the HAM results with the numerical results of [1]. Section 6 synthesis the concluding remarks.

2. Flow analysis

Consider the steady laminar flow of a second grade fluid over a radial stretching sheet. The sheet is in the plane $z=0$ and has stretched velocity proportional to the distance from the origin. The fluid occupies the half space $z>0$. The constitutive equation for the Cauchy stress in a second grade fluid is [2–4]

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{1}$$

where the first two kinematic tensors $\mathbf{A}_1$ and $\mathbf{A}_2$ are

$$\begin{aligned}\mathbf{A}_1 &= \nabla\mathbf{V} + (\nabla\mathbf{V})^\top,\\ \mathbf{A}_2 &= \frac{\mathrm{d}\mathbf{A}_1}{\mathrm{d}t} + \mathbf{A}_1(\nabla\mathbf{V}) + (\nabla\mathbf{V})^\top\mathbf{A}_1,\end{aligned} \tag{2}$$

in which $\mathbf{V}$ is the fluid velocity, $\mathrm{d}/\mathrm{d}t$ is the material derivative and α_1 and α_2 are respectively the viscoelasticity and cross-viscosity of the fluid. In order to satisfy the thermodynamic analysis, the conditions $\mu \geqslant 0$, $\alpha_1 \geqslant 0$, $\alpha_1+\alpha_2=0$ must hold.

For the mathematical modelling, we take cylindrical polar coordinate system (r, θ, z) when flow occurs under the rotational symmetry. Thus all physical quantities do not depend upon θ i.e. $\partial/\partial\theta = 0$ and azimuthal component (v) of velocity $V=(u,v,w)$ vanishes identically. With these facts the equations which govern the flow are

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0, \tag{3}$$

$$\rho\left(u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z}\right) = \frac{\partial T_{rr}}{\partial r} + \frac{\partial T_{rz}}{\partial z} + \frac{T_{rr}-T_{\theta\theta}}{r}, \tag{4}$$

$$\rho\left(u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}\right) = \frac{1}{r}\frac{\partial}{\partial r}(rT_{rz}) + \frac{\partial T_{zz}}{\partial z}, \tag{5}$$

where

$$\begin{aligned}T_{rr} = {}& -p + 2\mu\frac{\partial u}{\partial r} + 2\alpha_1\left[u\frac{\partial^2 u}{\partial r^2} + w\frac{\partial^2 u}{\partial r\partial z} + 2\left(\frac{\partial u}{\partial r}\right)^2\right.\\ &\left. + \frac{\partial w}{\partial r}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)\right] + \alpha_2\left[4\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)^2\right],\end{aligned} \tag{6}$$

$$T_{\theta\theta} = -p + 2\mu\frac{u}{r} + 2\alpha_1\left[\frac{u}{r}\frac{\partial u}{\partial r} + \frac{w}{r}\frac{\partial u}{\partial z} + \left(\frac{u}{r}\right)^2\right] + 4\alpha_2\left(\frac{u}{r}\right)^2, \tag{7}$$

$$\begin{aligned}T_{zz} = {}& -p + 2\mu\frac{\partial w}{\partial z} + 2\alpha_1\left[u\frac{\partial^2 w}{\partial r\partial z} + w\frac{\partial^2 w}{\partial z^2} + 2\left(\frac{\partial w}{\partial z}\right)^2\right.\\ &\left. + \frac{\partial u}{\partial z}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)\right] + \alpha_2\left[4\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)^2\right],\end{aligned} \tag{8}$$

$$\begin{aligned}T_{rz} = {}& \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) + \alpha_1\left[\left(u\frac{\partial}{\partial r} + w\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)\right.\\ &\left. + \frac{\partial u}{\partial r}\left(3\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) + \frac{\partial w}{\partial z}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)\right]\\ & + 2\alpha_2\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)\left(\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z}\right),\end{aligned} \tag{9}$$

and u and w are the velocities in the r- and z-directions, respectively and ρ is the fluid density.

The relevant boundary conditions are

$$\begin{aligned}&u = Br, \quad w = 0, \quad \text{at } z = 0,\\ &u \to 0 \quad \text{as } z \to \infty.\end{aligned} \tag{10}$$

Let us introduce the following dimensionless quantities

$$u = Brf'(\eta), \quad w = -2\sqrt{B\nu}f(\eta), \quad \eta = \sqrt{\frac{B}{\nu}}z, \quad K = \frac{B\alpha_1}{\mu}, \tag{11}$$

where the prime signifies differentiation with respect to η and B is the proportionality constant relating to the stretching of the sheet. The mass conservation equation is automatically satisfied and Eqs. (4) and (9) give the following expression for pressure

$$p = -2B\mu f^2 - 2B\mu f' + 2K\mu B\left[2ff'' + 3f'^2 + \frac{Br^2}{\nu}f''^2\right] + p_0, \tag{12}$$

where p_0 is a constant and the ordinary differential equation for velocities [1]

$$f''' - f'^2 + 2ff'' - K(f''^2 - 2f'f''' + 2ff^{iv}) = 0, \tag{13}$$

The boundary conditions (10) now become

$$\begin{aligned}&f = 0, \quad f' = 1 \quad \text{at } \eta = 0,\\ &f' \to 0 \quad \text{as } \eta \to \infty.\end{aligned} \tag{14}$$

The non-linear differential Eq. (13) governing the flow has to be solved subject to the conditions (14) by HAM.

2.1. Homotopy analytic solution

Here, we obtain the analytic and uniformly valid solution by HAM. For that we use the initial approximation

$$f_0(\eta) = 1 - e^{-\eta} \tag{15}$$

and the auxiliary linear operator

$$\mathscr{L}_1(f) = f''' + f'', \tag{16}$$

satisfying

$$\mathscr{L}_1[C_1\eta + C_2 + C_3 e^{-\eta}] = 0, \tag{17}$$

where C_i ($i = 1$–3) are constants.

The deformation problem at the zeroth-order satisfies

$$(1-p)\mathscr{L}_1[f(\eta,p) - f_0(\eta)] = p\hbar_1 \mathscr{N}_1[f(\eta,p)], \tag{18}$$

$$f(0,p) = 0, \quad f'(0,p) = 1, \quad f'(\infty,p) = 0, \tag{19}$$

where $\hbar_1$ and $p \in [0,1]$ are respectively the auxiliary and embedding parameters and

$$\mathscr{N}_1[f(\eta,p)] = \frac{\partial^3 f(\eta,p)}{\partial\eta^3} - \left(\frac{\partial f(\eta,p)}{\partial\eta}\right)^2 + 2f(\eta,p)\frac{\partial^2 f(\eta,p)}{\partial\eta^2} - K\left\{\left(\frac{\partial^2 f(\eta,p)}{\partial\eta^2}\right)^2 - 2\frac{\partial f(\eta,p)}{\partial\eta}\frac{\partial^3 f(\eta,p)}{\partial\eta^3} + 2f(\eta,p)\frac{\partial^4 f(\eta,p)}{\partial\eta^4}\right\}, \tag{20}$$

is the non-linear differential operator. For $p = 0$ and $p = 1$, we have

$$f(\eta,0) = f_0(\eta), \quad f(\eta,1) = f(\eta). \tag{21}$$

We note from above equation that the variation of p from 0 to 1 is continuous variation of $f(\eta,p)$ from $f_0(\eta)$ to $f(\eta)$. Due to Taylor's theorem and Eq. (21) we can write

$$f(\eta,p) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta)p^m, \tag{22}$$

in which

$$f_m(\eta) = \frac{1}{m!}\frac{\partial^m f(\eta,p)}{\partial p^m}\bigg|_{p=0}. \tag{23}$$

Clearly, the convergence of the series (22) depends on the auxiliary parameter $\hbar_1$. Assume that $\hbar_1$ is selected such that the series (22) is convergent at $p = 1$, then due to Eq. (21) we have

$$f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta). \tag{24}$$

Differentiating m-times the zeroth-order deformation Eq. (18) with respect to p and then dividing them by $m!$ and finally setting $p = 0$ we have the mth-order deformation problem

$$\mathscr{L}_1[f_m(\eta) - \chi_m f_{m-1}(\eta)] = \hbar_1 \mathscr{R}_m^f(\eta), \tag{25}$$

$$f_m(0) = f_m'(0) = f_m'(\infty) = 0, \tag{26}$$

$$\mathscr{R}_m^f(\eta) = f_{m-1}'''(\eta) + \sum_{k=0}^{m-1}\left[2f_{m-1-k}f_k'' - f_{m-1-k}'f_k' - K\left(f_{m-1-k}''\,f_k'' - 2f_{m-1-k}'\,f_k''' + 2f_{m-1-k}\,f_k^{iv}\right)\right], \tag{27}$$

$$\chi_m = \begin{cases} 0, & m \leqslant 1, \\ 1, & m > 1. \end{cases} \tag{28}$$

For the solution of the mth-order problem, we use the symbolic computation software MATHEMATICA up to first few order of approximation. It is found that the solution of this problem is

$$f_m(\eta) = \sum_{n=0}^{m+1}\sum_{k=0}^{m+1-n} a_{m,n}^k \eta^k e^{-n\eta}, \quad m \geqslant 0. \tag{29}$$

In order to obtain the recurrence formulae for the coefficients $a_{m,n}^q$ of $f_m(\eta)$, we substitute Eq. (29) in Eq. (25) and obtain for $m \geqslant 1$, $0 \leqslant n \leqslant m+1$ and $0 \leqslant q \leqslant m+1-n$

$$a_{m,0}^0 = \chi_m\chi_{2m+1}a_{m-1,0}^0 - \sum_{q=0}^{m}\Psi_{m,1}^q\mu_{1,1}^q - \sum_{n=2}^{m+1}\left[(n-1)\Psi_{m,n}^q\mu_{n,0}^0 + \sum_{q=1}^{m+1-n}\Psi_{m,n}^q(n\mu_{n,0}^q - \mu_{n,1}^q)\right], \tag{30}$$

$$a_{m,0}^k = \chi_m\chi_{2m+1-k}a_{m-1,0}^k, \quad 1 \leqslant k \leqslant m+1, \tag{31}$$

$$a_{m,1}^0 = \chi_m\chi_{2m}a_{m-1,1}^0 + \sum_{q=0}^{m}\Psi_{m,1}^q\mu_{1,1}^q + \sum_{n=2}^{m+1}\left\{n\Psi_{m,n}^q\mu_{n,0}^0 + \sum_{q=1}^{m+1-n}\Psi_{m,n}^q(n\mu_{n,0}^q - \mu_{n,1}^q)\right\}, \tag{32}$$

$$a_{m,1}^k = \chi_m\chi_{2m-k}a_{m-1,1}^k + \sum_{q=k-1}^{m}\Psi_{m,1}^q\mu_{1,k}^q, \quad 1 \leqslant k \leqslant m+1, \tag{33}$$

$$a_{m,n}^k = \chi_m\chi_{2m+1-n-k}a_{m-1,n}^k + \sum_{q=k}^{m+1-n}\Psi_{m,n}^q\mu_{n,k}^q, \quad 2 \leqslant n \leqslant m+1, \quad 0 \leqslant k \leqslant m+1-n, \tag{34}$$

where

$$\mu_{1,k}^q = \frac{(q-k+2)q!}{k!}, \quad 0 \leqslant k \leqslant q+1, \quad q \geqslant 0, \tag{35}$$

$$\mu_{n,k}^q = \frac{q!}{k!(n-1)^{q-k+1}} \times \left[1 - \left(1 - \frac{1}{n}\right)^{q-k+1}\left\{(q-k+2). - (q-k+1)\left(1 - \frac{1}{n}\right)\right\}\right], \quad 0 \leqslant k \leqslant q+1-n, \ q \geqslant 0, \ n \geqslant 2. \tag{36}$$

The coefficient $\Psi^q_{m,n}$ is defined by

$$\Psi^q_{m,n} = \hbar_1[\chi_{m-n-q+2} d^q_{m-1,n} + \{2\delta^q_{m,n} - \Delta^q_{m,n} - K(\Lambda^q_{m,n} - 2\Gamma^q_{m,n} + 2\Omega^q_{m,n})\}]. \quad (37)$$

where the coefficients $\delta^q_{m,n}$, $\Delta^q_{m,n}$, $\Lambda^q_{m,n}$, $\Gamma^q_{m,n}$ and $\Omega^q_{m,n}$ for $m \geqslant 1$, $0 \leqslant n \leqslant m+1$, $0 \leqslant q \leqslant m+1-n$ are

$$\delta^q_{m,n} = \sum_{k=0}^{m-1} \sum_{j=\max\{0,n-m+k\}}^{\min\{n,k+1\}} \sum_{i=\max\{0,q-m+k+n-j\}}^{\min\{q,k+1-j\}} c^i_{k,j} a^{q-i}_{m-1-k,n-j}, \quad (38)$$

$$\Delta^q_{m,n} = \sum_{k=0}^{m-1} \sum_{j=\max\{0,n-m+k\}}^{\min\{n,k+1\}} \sum_{i=\max\{0,q-m+k+n-j\}}^{\min\{q,k+1-j\}} b^i_{k,j} b^{q-i}_{m-1-k,n-j}, \quad (39)$$

$$\Lambda^q_{m,n} = \sum_{k=0}^{m-1} \sum_{j=\max\{0,n-m+k\}}^{\min\{n,k+1\}} \sum_{i=\max\{0,q-m+k+n-j\}}^{\min\{q,k+1-j\}} c^i_{k,j} c^{q-i}_{m-1-k,n-j}, \quad (40)$$

$$\Gamma^q_{m,n} = \sum_{k=0}^{m-1} \sum_{j=\max\{0,n-m+k\}}^{\min\{n,k+1\}} \sum_{i=\max\{0,q-m+k+n-j\}}^{\min\{q,k+1-j\}} d^i_{k,j} b^{q-i}_{m-1-k,n-j}, \quad (41)$$

$$\Omega^q_{m,n} = \sum_{k=0}^{m-1} \sum_{j=\max\{0,n-m+k\}}^{\min\{n,k+1\}} \sum_{i=\max\{0,q-m+k+n-j\}}^{\min\{q,k+1-j\}} e^i_{k,j} a^{q-i}_{m-1-k,n-j}, \quad (42)$$

and the coefficients $b^k_{m,n}$, $c^k_{m,n}$, $d^k_{m,n}$ and $e^k_{m,n}$ are

$$b^k_{m,n} = (k+1)a^{k+1}_{m,n} - n a^k_{m,n}, \quad (43)$$

$$c^k_{m,n} = (k+1)b^{k+1}_{m,n} - n b^k_{m,n}, \quad (44)$$

$$d^k_{m,n} = (k+1)c^{k+1}_{m,n} - n c^k_{m,n}, \quad (45)$$

$$e^k_{m,n} = (k+1)d^{k+1}_{m,n} - n d^k_{m,n}. \quad (46)$$

For the detailed procedure of deriving the above relations the reader is referred to [11]. With the above recurrence formulae, we can calculate all coefficients $a^k_{m,n}$ using only the first two

$$a^0_{0,0} = 1, \quad a^0_{0,1} = -1, \quad (47)$$

given by the initial guess approximations for the function $f(\eta)$ in Eq. (15). The corresponding Mth-order approximation of Eqs. (13) and (14) is then given by

$$\sum_{m=0}^{M} f_m(\eta) = \sum_{m=0}^{M} a^0_{m,0} + \sum_{n=1}^{M+1} e^{-n\eta} \left(\sum_{m=n-1}^{M} \sum_{k=0}^{m+1-n} a^k_{m,n} \eta^k \right). \quad (48)$$

The explicit, totally analytic solution of the present flow is

$$f(\eta) = \sum_{m=0}^{\infty} f_m(\eta) = \lim_{M\to\infty} \left[\sum_{m=0}^{M} a^0_{m,0} + \sum_{n=1}^{M+1} e^{-n\eta} \left(\sum_{m=n-1}^{M} \sum_{k=0}^{m+1-n} a^k_{m,n} \eta^k \right) \right]. \quad (49)$$

2.2. Skin friction

The expression for shear stress τ_w on the surface of the stretching sheet is defined as

$$\tau_w = T_{rz}|_{z=0} \quad (50)$$

and the local skin friction coefficient or frictional drag coefficient is

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho(Br)^2}. \quad (51)$$

In terms of dimensionless variables we have

$$C_f = 2Re_r^{-1/2}[f''(\eta) + 2K\{2f'(\eta)f''(\eta) - f(\eta)f'''(\eta)\}]|_{\eta=0}, \quad (52)$$

where $Re_r = Br^2/\nu$ is the local Reynolds number based on the length scale r.

3. Heat transfer analysis

The energy equation, corresponding to the axisymmetric flow of a second grade fluid is

$$\begin{aligned}
\rho c_p\left(u\frac{\partial T}{\partial r} + w\frac{\partial T}{\partial z}\right)
&= k\left(\frac{\partial^2 T}{\partial r^2} + \frac{T_r}{r} + \frac{\partial^2 T}{\partial z^2}\right) + \mu\left\{\frac{2u^2}{r^2} + 2\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 \right.\\
&\left. +2\frac{\partial w}{\partial r}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) + 2\left(\frac{\partial w}{\partial z}\right)^2\right\} + \alpha_1\left[\frac{2u^3}{r^3} + \frac{2u^2}{r}\frac{\partial u}{\partial r}\right.\\
&+4\left(\frac{\partial u}{\partial r}\right)^3 + 2u\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r^2} + 2w\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r\partial z} + \frac{2uw}{r^2}\frac{\partial u}{\partial z}\\
&+u\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r\partial z} + 3\frac{\partial u}{\partial r}\left(\frac{\partial u}{\partial z}\right)^2 + w\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2} + u\frac{\partial^2 u}{\partial r\partial z}\frac{\partial w}{\partial r}\\
&+6\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r} + w\frac{\partial^2 u}{\partial z^2}\frac{\partial w}{\partial r} + 3\frac{\partial u}{\partial r}\left(\frac{\partial w}{\partial r}\right)^2 + u\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r^2}\\
&+u\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2} + w\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r\partial z} + w\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r\partial z} + 3\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial w}{\partial z}\\
&+4\left(\frac{\partial w}{\partial z}\right)^3 + 6\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\frac{\partial w}{\partial z} + 3\frac{\partial w}{\partial z}\left(\frac{\partial w}{\partial r}\right)^2 + 2u\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z}\\
&\left.+2w\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2}\right] + \alpha_2\left[\frac{4u^3}{r^3} + 4\left(\frac{\partial u}{\partial r}\right)^3 + 3\frac{\partial u}{\partial r}\left(\frac{\partial u}{\partial z}\right)^2\right.\\
&+6\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r} + 4\left(\frac{\partial w}{\partial z}\right)^3 + 3\frac{\partial u}{\partial r}\left(\frac{\partial w}{\partial r}\right)^2 + 3\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial w}{\partial z}\\
&\left.+6\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\frac{\partial w}{\partial z} + 3\frac{\partial w}{\partial z}\left(\frac{\partial w}{\partial r}\right)^2\right]. \quad (53)
\end{aligned}$$

In above equation T is the temperature, c_p is the specific heat and k is the thermal conductivity. The boundary conditions depend on the heating process.

3.1. The prescribed surface temperature (PST case)

Here

$$T = T_w = T_\infty + A\left(\frac{r}{l}\right)^2 \quad \text{at } z = 0, \qquad T \to T_\infty \quad \text{as } z \to \infty, \quad (54)$$

where A and l are constants. Defining

$$\theta(\eta)=\frac{T-T_\infty}{T_w-T_\infty}, \tag{55}$$

the governing problem is of the following form

$$\theta''+\frac{4}{\delta}\theta+Pr\left[(2f\theta'-2f'\theta)+E\left\{f''^2+\frac{12}{\delta}f'^2+K\left(f'f''^2-2ff''f'''-\frac{24}{\delta}ff'f''\right)\right\}\right]=0, \tag{56}$$

$$\theta=1 \quad \text{at } \eta=0,$$
$$\theta\to 0 \quad \text{as } \eta\to\infty, \tag{57}$$

where $Pr=\mu c_p/k$, $E=B^2l^2/c_pA$ and $\delta=Br^2/\nu$ are the Prandtl number, Eckert number and the local Reynold number, respectively.

3.1.1. HAM solution

Here the initial guess approximation of $\theta(\eta)$ and the corresponding auxiliary operator are respectively selected as

$$\theta_0(\eta)=e^{-\eta}, \tag{58}$$
$$\mathscr{L}_2(f)=f''+f', \tag{59}$$

where

$$\mathscr{L}_2[C_4+C_5e^{-\eta}]=0, \tag{60}$$

and C_4 and C_5 are arbitrary constants.

The zeroth-order problem is

$$(1-p)\mathscr{L}_2[\theta(\eta,p)-\theta_0(\eta)]=p\hbar_2\mathscr{N}_2[\theta(\eta,p),f(\eta,p)], \tag{61}$$
$$\theta(0,p)=1,\quad \theta(\infty,p)=0, \tag{62}$$

with the non-linear differential operator $\mathscr{N}_2$ of the following form:

$$\mathscr{N}_2[\theta(\eta,p),f(\eta,p)]=\frac{\partial^2\theta(\eta,p)}{\partial\eta^2}+\frac{4}{\delta}\theta(\eta,p)+Pr\left[2f(\eta,p)\frac{\partial\theta(\eta,p)}{\partial\eta}-2\frac{\partial f(\eta,p)}{\partial\eta}\theta(\eta,p)+E\left\{\left(\frac{\partial^2 f(\eta,p)}{\partial\eta^2}\right)^2+\frac{12}{\delta}\left(\frac{\partial f(\eta,p)}{\partial\eta}\right)^2+K\frac{\partial^2 f(\eta,p)}{\partial\eta^2}\left(\frac{\partial f(\eta,p)}{\partial\eta}\frac{\partial^2 f(\eta,p)}{\partial\eta^2}-2f(\eta,p)\frac{\partial^3 f(\eta,p)}{\partial\eta^3}-\frac{24}{\delta}f(\eta,p)\frac{\partial f(\eta,p)}{\partial\eta}\right)\right\}\right], \tag{63}$$

in which $\hbar_2$ is auxiliary nonzero parameter. For $p=0$ and $p=1$, we have

$$\theta(\eta,0)=\theta_0(\eta),\quad \theta(\eta,1)=\theta(\eta). \tag{64}$$

Obviously as p increases from 0 to 1, $\theta(\eta,p)$ varies from $\theta_0(\eta)$ to $\theta(\eta)$. By Taylor's theorem and Eq. (64), we have

$$\theta(\eta,p)=\theta_0(\eta)+\sum_{m=1}^{\infty}\theta_m(\eta)p^m, \tag{65}$$

where

$$\theta_m(\eta)=\frac{1}{m!}\left.\frac{\partial^m\theta(\eta,p)}{\partial p^m}\right|_{p=0} \tag{66}$$

and convergence of series (65) depends on $\hbar_2$. Assume that $\hbar_2$ is selected such that the series (65) is convergent at $p=1$, then due to Eq. (64) we can write

$$\theta(\eta)=\theta_0(\eta)+\sum_{m=1}^{\infty}\theta_m(\eta). \tag{67}$$

The mth-order deformation problem is

$$\mathscr{L}_2[\theta_m(\eta)-\chi_m\theta_{m-1}(\eta)]=\hbar_2\mathscr{R}_m^{\theta}(\eta), \tag{68}$$
$$\theta_m(0)=\theta_m(\infty)=0, \tag{69}$$

where

$$\mathscr{R}_m^{\theta}(\eta)=\theta''_{m-1}(\eta)+\frac{4}{\delta}\theta_{m-1}(\eta)+Pr\sum_{k=0}^{m-1}\left[2f_{m-1-k}\theta'_k-2f'_{m-1-k}\theta_k+E\left\{f''_{m-1-k}f''_k+\frac{12}{\delta}f'_{m-1-k}f'_k+K\left(f'_{m-1-k}\sum_{l=0}^{k}f''_{k-l}f''_l-2f_{m-1-k}\sum_{l=0}^{k}\left(f''_{k-l}f'''_l+\frac{12}{\delta}f'_{k-l}f''_l\right)\right)\right\}\right]. \tag{70}$$

The solution of the above problem can be expressed as

$$\theta_m(\eta)=\sum_{n=0}^{m+2}\sum_{q=0}^{m+2-n}A_{m,n}^q\eta^q e^{-n\eta},\quad m\geqslant 0. \tag{71}$$

Upon making use of the expression given in Eq. (71) into Eq. (68) yields the following recurrence formulae for the coefficients $A_{m,n}^q$ of $\theta_m(\eta)$:

for $m\geqslant 1$, $0\leqslant n\leqslant m+2$ and $0\leqslant q\leqslant m+2-n$:

$$A_{m,1}^0=\chi_m\chi_{2m}A_{m-1,1}^0-\sum_{n=2}^{m+2}\sum_{q=0}^{m+2-n}\Theta_{m,n}^q\nu_{n,0}^q, \tag{72}$$

$$A_{m,1}^k=\chi_m\chi_{2m-k}A_{m-1,1}^k-\sum_{q=k-1}^{m+1}\Theta_{m,1}^q\nu_{1,k}^q,\quad 1\leqslant k\leqslant m+1, \tag{73}$$

$$A_{m,n}^k=\chi_m\chi_{2m+1-n-k}A_{m-1,n}^k-\sum_{q=k}^{m+2-n}\Theta_{m,n}^q\nu_{n,k}^q,\quad 2\leqslant n\leqslant m+2,\ 0\leqslant k\leqslant m+2-n, \tag{74}$$

$$\nu_{1,k}^q=\frac{q!}{k!},\quad 0\leqslant k\leqslant q+2,\ q\geqslant 0, \tag{75}$$

$$\nu_{n,k}^q=\sum_{p=0}^{q-k}\frac{q!}{k!(n-1)^{q-p+1}(n)^{p+1}},\quad 0\leqslant k\leqslant q+2-n,\ q\geqslant 0,\ n\geqslant 2, \tag{76}$$

$$\Theta_{m,n}^q=\hbar_2\left[\chi_{m+3-n-q}\left\{C_{m-1,n}^q+\frac{4}{\delta}A_{m-1,n}^q+PrE\left(\Lambda_{m,n}^q+\frac{12}{\delta}\Delta_{m,n}^q\right)\right\}+\left\{Pr(2\Pi_{m,n}^q-2\lambda_{m,n}^q)+PrEK\left(\gamma_{m,n}^q-2\kappa_{m,n}^q-\frac{24}{\delta}\sigma_{m,n}^q\right)\right\}\right]. \tag{77}$$

The coefficients $\Pi^q_{m,n}$, $\lambda^q_{m,n}$, $\gamma^q_{m,n}$, $\kappa^q_{m,n}$ and $\sigma^q_{m,n}$ where $m \geqslant 1$, $0 \leqslant n \leqslant m+2$, $0 \leqslant q \leqslant m+2-n$ are defined by

$$\Pi^q_{m,n} = \sum_{k=0}^{m-1} \sum_{j=\max\{0,n-m+k\}}^{\min\{n,k+2\}} \sum_{i=\max\{0,q-m+k+n-j\}}^{\min\{q,k+2-j\}} B^i_{k,j} a^{q-i}_{m-1-k,n-j}, \tag{78}$$

$$\lambda^q_{m,n} = \sum_{k=0}^{m-1} \sum_{j=\max\{0,n-m+k\}}^{\min\{n,k+2\}} \sum_{i=\max\{0,q-m+k+n-j\}}^{\min\{q,k+2-j\}} A^i_{k,j} b^{q-i}_{m-1-k,n-j}, \tag{79}$$

$$\gamma^q_{m,n} = \sum_{k=0}^{m-1} \sum_{l=0}^{k} \sum_{r=\max\{0,n-m+k\}}^{\min\{n,k+2\}} \sum_{s=\max\{0,q-m+k+n-r\}}^{\min\{q,k+2-r\}} \sum_{j=\max\{0,r-k+l-1\}}^{\min\{p,l+1\}} \times \sum_{i=\max\{0,s-k+l+r-j+1\}}^{\min\{t,l+1-j\}} c^i_{l,j} c^{t-i}_{k-l,p-j} b^{q-t}_{m-1-k,n-p}, \tag{80}$$

$$\kappa^q_{m,n} = \sum_{k=0}^{m-1} \sum_{l=0}^{k} \sum_{r=\max\{0,n-m+k\}}^{\min\{n,k+2\}} \sum_{s=\max\{0,q-m+k+n-r\}}^{\min\{q,k+2-r\}} \sum_{j=\max\{0,r-k+l-1\}}^{\min\{p,l+1\}} \times \sum_{i=\max\{0,s-k+l+r-j+1\}}^{\min\{t,l+1-j\}} d^i_{l,j} c^{t-i}_{k-l,p-j} a^{q-t}_{m-1-k,n-p}, \tag{81}$$

$$\sigma^q_{m,n} = \sum_{k=0}^{m-1} \sum_{l=0}^{k} \sum_{r=\max\{0,n-m+k\}}^{\min\{n,k+2\}} \sum_{s=\max\{0,q-m+k+n-r\}}^{\min\{q,k+2-r\}} \sum_{j=\max\{0,r-k+l-1\}}^{\min\{p,l+1\}} \times \sum_{i=\max\{0,s-k+l+r-j+1\}}^{\min\{t,l+1-j\}} c^i_{l,j} b^{t-i}_{k-l,p-j} a^{q-t}_{m-1-k,n-p}, \tag{82}$$

where the related coefficients $B^k_{m,n}$ and $C^k_{m,n}$ are given by

$$B^k_{m,n} = (k+1)A^{k+1}_{m,n} - nA^k_{m,n}, \tag{83}$$

$$C^k_{m,n} = (k+1)B^{k+1}_{m,n} - nB^k_{m,n}. \tag{84}$$

Using the above recurrence formulae, we can calculate all coefficients $A^k_{m,n}$ using only the first two

$$A^0_{0,0} = 0, \quad A^0_{0,1} = 1, \tag{85}$$

given by the initial guess approximations for the function $\theta(\eta)$ in Eq. (58). The corresponding Mth-order approximation of Eqs. (56) and (57) is

$$\sum_{m=0}^{M} \theta_m(\eta) = \sum_{n=1}^{M+2} e^{-n\eta} \left(\sum_{m=n-1}^{M+1} \sum_{k=0}^{m+2-n} A^k_{m,n} \eta^k \right). \tag{86}$$

Therefore the explicit, totally analytic solution of the heat transfer in the PST case

$$\theta(\eta) = \sum_{m=0}^{\infty} \theta_m(\eta) = \lim_{M\to\infty} \left[\sum_{n=1}^{M+2} e^{-n\eta} \left(\sum_{m=n-1}^{M+1} \sum_{k=0}^{m+2-n} A^k_{m,n} \eta^k \right) \right] \tag{87}$$

and the dimensionless temperature gradient at the wall is given by

$$\theta'(0) = \sum_{m=0}^{\infty} \theta'_m(0) = \lim_{M\to\infty} \left[\sum_{n=1}^{M+2} \sum_{m=n-1}^{M+1} (A^1_{m,n} - A^0_{m,n}) \right]. \tag{88}$$

The dimensionless heat transfer rate at the wall, characterized by the Nusselt number Nu, is given by

$$Nu = \frac{-k\frac{\partial T}{\partial z}\big|_{z=0}}{k(T_w - T_\infty)} r = -Re_r^{1/2}\theta'(0) \tag{89}$$

and the local heat flux can be expressed as

$$q_w = -k\frac{\partial T}{\partial z}\bigg|_{z=0} = -kA\left(\frac{r}{l}\right)^2 \sqrt{\frac{B}{\nu}}\theta'(0). \tag{90}$$

The expressions in Eqs. (87) and (89) are evaluated for the different values of the emerging parameters and are discussed. We will now discuss the case of the prescribed heat flux in the next subsection.

3.2. The prescribed surface heat flux (PHF case)

The appropriate boundary conditions are

$$-k\frac{\partial T}{\partial z} = q_w = D\left(\frac{r}{l}\right)^2 \quad \text{at } z = 0, \tag{91}$$

$$T \to T_\infty \quad \text{as } z \to \infty. \tag{92}$$

Taking

$$T - T_\infty = \frac{D}{k}\left(\frac{r}{l}\right)^2 \sqrt{\frac{\nu}{B}} g(\eta), \tag{93}$$

the resulting problem consists of Eq. (56) with the following boundary conditions:

$$g'(\eta) = -1, \quad \text{at } \eta = 0, \qquad g(\eta) = 0 \quad \text{as } \eta \to \infty. \tag{94}$$

where the Eckert number here is defined as

$$E = \frac{kB^2 l^2}{Dc_p}\sqrt{\frac{B}{\nu}}. \tag{95}$$

3.2.1. HAM solution

We note that solution here is the same as in the previous subsection except that now the recurrence relation in Eq. (72) is

$$A^0_{m,1} = \chi_m \chi_{2m} A^0_{m-1,1} - \sum_{q=0}^{m+2} \Theta^q_{m,1} v^q_{1,1} - \sum_{n=2}^{m+2} \left[n\Theta^0_{m,n} v^0_{n,0} + \sum_{q=1}^{m+2-n} \Theta^q_{m,n}(nv^q_{n,0} - v^q_{n,1}) \right]. \tag{96}$$

The corresponding Mth-order approximation of Eqs. (56) and (94) is

$$\sum_{m=0}^{M} g_m(\eta) = \sum_{n=1}^{M+2} e^{-n\eta} \left(\sum_{m=n-1}^{M+1} \sum_{k=0}^{m+2-n} A^k_{m,n} \eta^k \right) \tag{97}$$

and totally analytic solution of the heat transfer in the PHF case is

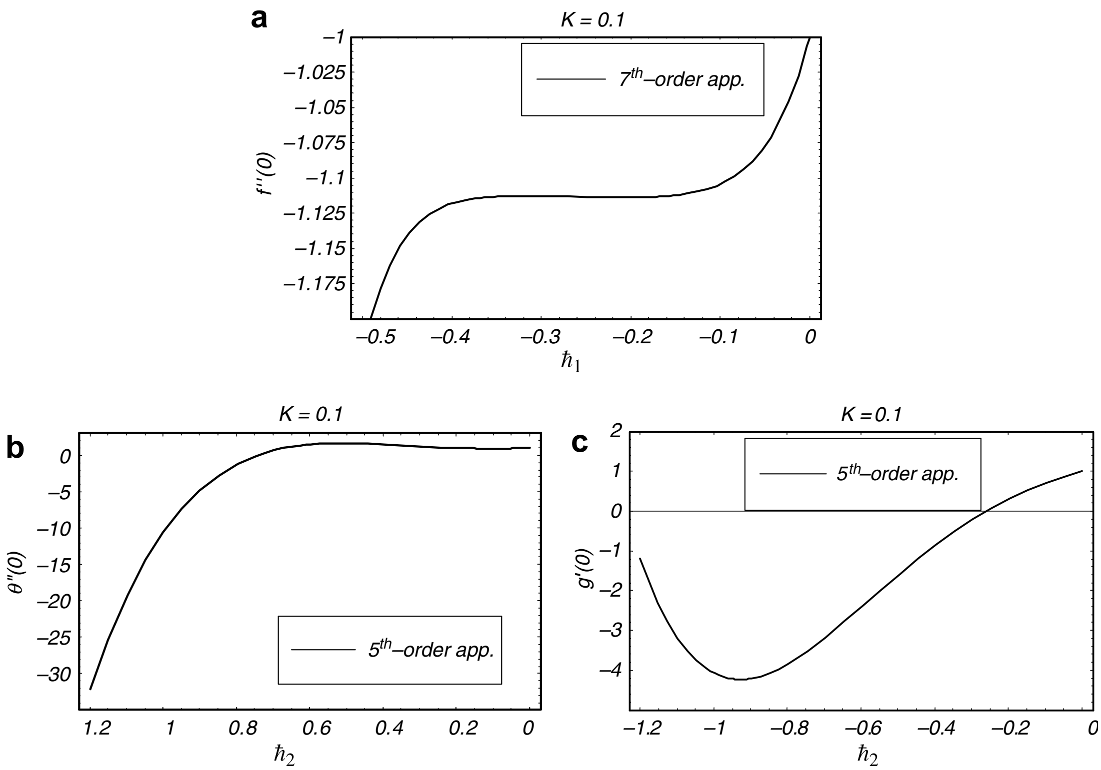

Fig. 1. $\hbar$-Curves are plotted for the functions f, θ and g. (a) Flow analysis, (b) PST case, and (c) PHF case.

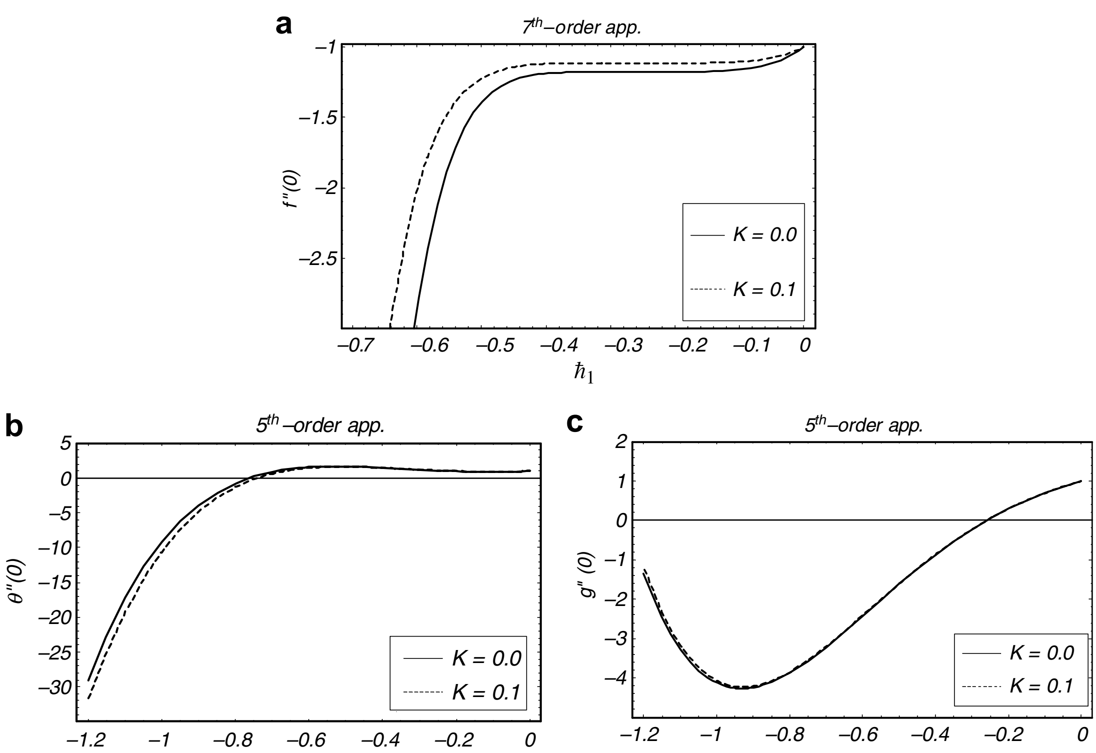

Fig. 2. Variation of $\hbar_{1,2}$ with increase in parameter K. (a) Flow analysis, (b) PST case, and (c) PHF case.

$$g(\eta) = \sum_{m=0}^{\infty} g_m(\eta) = \lim_{M\to\infty}\left[\sum_{n=1}^{M+2} e^{-n\eta}\left(\sum_{m=n-1}^{M+1}\sum_{k=0}^{m+2-n} A_{m,n}^{k}\eta^{k}\right)\right]. \tag{98}$$

The wall temperature T_w is obtained from Eq. (93) as

$$T - T_{\infty} = \frac{D}{k}\left(\frac{r}{l}\right)^2\sqrt{\frac{\nu}{B}}g(0). \tag{99}$$

4. The convergence of the solution

The explicit, analytic expressions (49), (87) and (98) contain two auxiliary parameters $\hbar_1$ and $\hbar_2$. As pointed out by Liao [6], the convergence region and rate of approximations given by the homotopy analysis method are strongly dependent upon these auxiliary parameters. In Fig. 1(a)–(c) the $\hbar$-curves are plotted to see the range of admissible values for the parameters $\hbar_1$ and $\hbar_2$. Fig. 1(a)–(c) display that the range for the admissible values for $\hbar_1$ and $\hbar_2$ is $-1 \leqslant \hbar_1, \hbar_2 < 0$. Also the series given in Eqs. (49), (87) and (98) converges

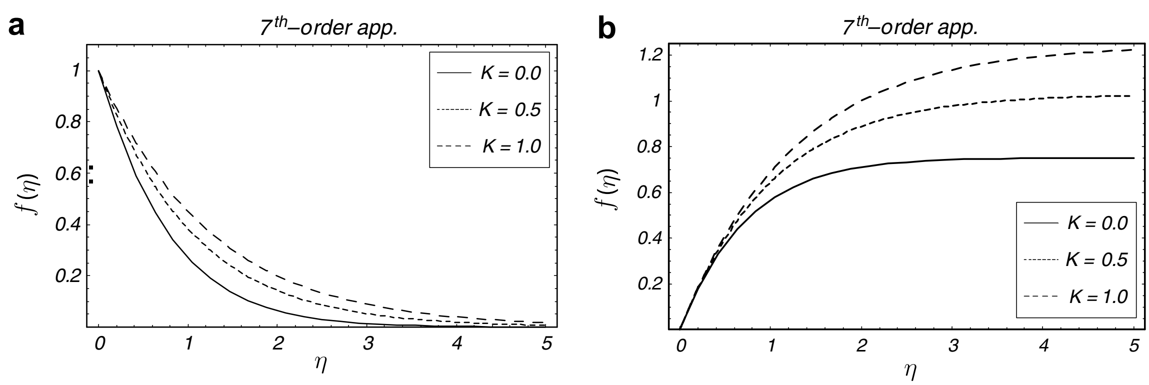

Fig. 3. Variation of the dimensionless velocity fields f' and f with increasing second grade parameter K. (a) $f'(\eta)$ and (b) $f(\eta)$.

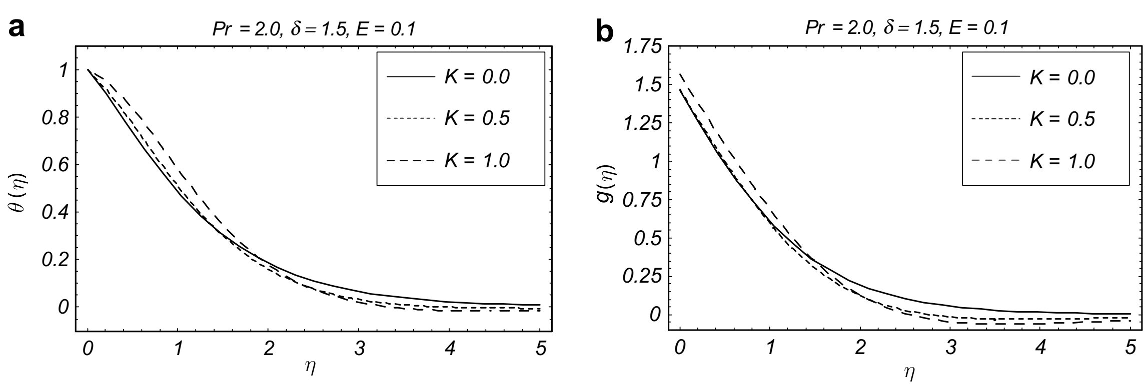

Fig. 4. Variation of the dimensionless temperature profiles θ and g with increasing second grade parameter K. (a) $\theta(\eta)$, PST case and (b) $g(\eta)$, PHF case.

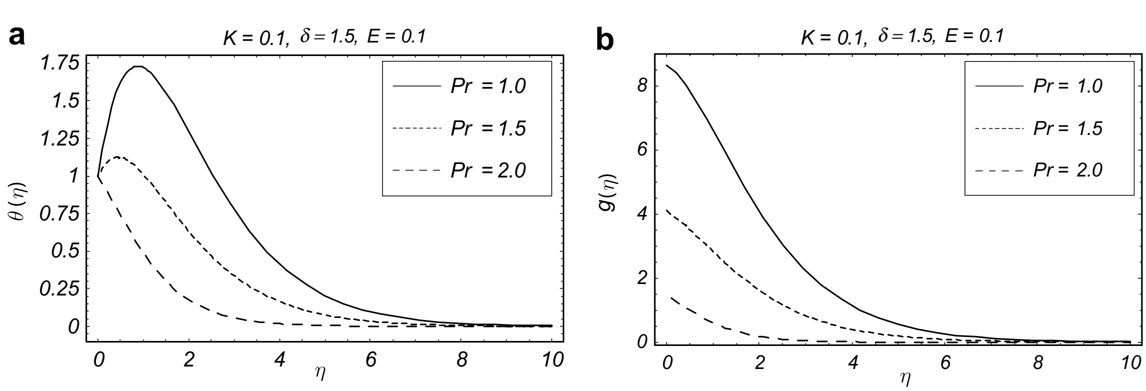

Fig. 5. Variation of the dimensionless temperature profiles θ and g with increasing Prandtl number Pr. (a) $\theta(\eta)$, PST case and (b) $g(\eta)$, PHF case.

in the whole region of η, when $\hbar_1 = -0.3$, $\hbar_2 = -0.6$ for the PST case and $\hbar_2 = -0.9$ for the PHF case.

Further Fig. 1(a) shows that range for the admissible values for $\hbar_1$ increases with increase in order of approximation. It is also found that the series (49) of $f(\eta)$ converges faster than that of the $\theta(\eta)$ and $g(\eta)$. This is due to the fact that the non-linearity in the later case is stronger than the former. Fig. 2 indicates the $\hbar$-curves for two different values of the second grade parameter K. It is observed that the interval for $\hbar_1$ increases by increasing K. Whereas in the case of $\hbar_2$ it is going towards zero as we increase the parameter K.

5. Results and discussion

This section deals with the variations of K, δ, Pr and E. For this purpose Figs. 3–7 have been sketched. In order to see the variation of second grade parameter K on velocity components u and w, the main emphasis has been given to plot the graphs for $f'(\eta)$ and $f(\eta)$ in Fig. 3. The graphs for the variation of K, δ, Pr and E on the temperature are shown in Figs. 4–7. In these figures, $\theta(\eta)$ is the temperature variation that corresponds to the PST case and $g(\eta)$ is the temperature variation for the PHF case. Moreover, the variations of K on the skin friction coefficient have been listed in Tables 1 and 2 have been prepared to show the variation of Pr and E on the Nusselt number. In Table 3 we have given the comparison of our results with those given in Ref. [1]. From the present study, it is concluded that:

- The r-component of velocity and boundary layer thickness increases by increasing K.

Table 1
Values of the skin friction coefficient $C_f Re_r^{1/2}$ for $\hbar_1 = -0.3$

K	$C_f Re_r^{1/2}$
0.0	−2.35591
0.1	−3.11952
0.2	−3.80538
0.3	−4.42764
0.4	−4.99863

Table 2
Values of the Nusselt number $-Re_r^{-1/2}\theta'(0)$ for $K = 0.1$, $\hbar_2 = -0.75$ and $\hbar_1 = -0.3$

Pr	$E = 0.0$	$E = 0.1$	$E = 0.2$	$E = 0.3$
0.5	−2.61875	−2.85423	−3.08971	−3.32518
1.0	−0.61250	−1.01080	−1.40910	−1.80740
1.5	0.83125	0.34278	−0.14568	−0.63415
2.0	1.71250	1.20652	0.70055	0.19456
2.5	2.03125	1.58042	1.12955	0.67877

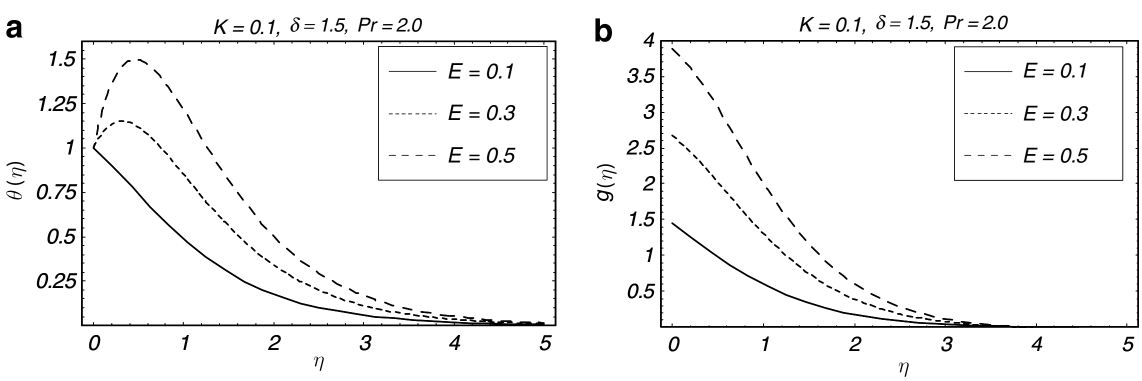

Fig. 6. Variation of the dimensionless temperature profiles θ and g with increasing Eckert number E. (a) $\theta(\eta)$, PST case and (b) $g(\eta)$, PHF case.

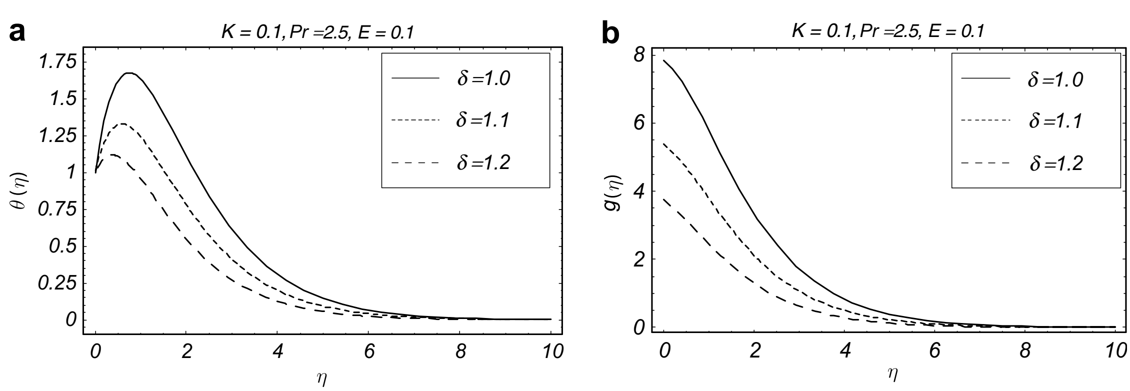

Fig. 7. Variation of the dimensionless temperature profiles θ and g with increasing parameter δ. (a) $\theta(\eta)$, PST case and (b) $g(\eta)$, PHF case.

Table 3
Illustrating the variation of $-f''(0)$ with K using: (i) exact numerical integration, (ii) perturbation solution for small K, (iii) asymptotic solution for large K, (iv) approximate solution for any K and (v) HAM solution for any K when $\hbar_1 = -0.25$

K	Exact [1]	Perturbation [1]	Asymtotic [1]	Approximate [1]	HAM
0.0	1.17372	1.17372		1.15470	1.17559
0.05	1.14241	1.14198		1.11906	1.14389
0.1	1.11221	1.11024		1.08641	1.11369
0.2	1.05603	1.04677		1.02864	1.05764
0.3	1.00580	0.98329		0.97899	1.00716
0.4	0.96115	0.91982		0.93579	0.96195
0.5	0.92140	0.85634		0.89776	0.92167
0.6	0.88586	0.79286		0.86396	0.88597
0.7	0.85393	0.72938		0.83366	0.85446
0.8	0.82507	0.66591	0.66737	0.80632	0.82675
0.9	0.79885	0.60243	0.67642	0.78148	0.80241
1.0	0.77491	0.53895	0.67753	0.75879	0.78098
1.1	0.75295	0.47548	0.67395	0.73795	0.76200
1.2	0.73272	0.41200	0.66756	0.71873	0.74497
1.6	0.66535		0.63124	0.65451	0.68550
2.0	0.61355		0.59310	0.60486	0.61115

- The z-component of velocity increases and boundary layer thickness decreases for large K.
- An increase in the value of K increases the temperature to a value $\eta = 1.8$ and then decreases the temperature. But the thermal boundary layer thickness increases by increasing K.
- By increasing the Prandtl number Pr, the temperature is found to decrease.
- The influence of E on the temperature is quite opposite of Pr.
- The effect of the parameter δ is similar to that of Pr.
- The magnitude of skin friction coefficient increases by increasing K.
- The Nusselt number decreases by increasing E and fixed Pr.
- The results obtained by HAM in case of velocity are in agreement with the numerical results of Ariel [1].

6. Concluding remarks

In this paper, we have considered a problem concerning the axisymmetric flow and heat transfer analysis of the second grade fluid. The solution of the problem is obtained by using HAM. To carry out heat transfer analysis, the energy equation has been solved for the prescribed surface temperature and heat flux cases. Analytical solutions for the velocity and temperature distributions are obtained using an analytical technique, namely the homotopy analysis method [5,6]. The convergence of the results are shown. The results are presented graphically and the effects of the emerging parameters are seen. The skin friction coefficient and the Nusselt number are tabulated. The results are also compared with the numerical results already presented in the literature [1] and found in excellent agreement.

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